International Conference on
Non-Euclidean Geometry and its applications.
5 – 9 July 2010, Kluj-Napoca (Kolozvár), ROMANIA

E.M. Ovsiyuk, V.M. Red'kov.

EXACT SPECTRUM FOR QUANTUM OSCILLATOR IN SPACES OF CONSTANT CURVATURE FROM WKB-QUANTIZATION

Institute of Physics
National Academy of Sciences of Belarus

Quantum-mechanical WKB-method is elaborated for the known quantum oscillator problem in curved 3-spaces models Euclid, Riemann, and Lobachevsky E_3, H_3, S_3 in the framework of the complex variable function theory. Generalized generally covariant Schrödinger equation is considered. In all three space models, exact energy levels are found with the help of constructing special formal WKB-sieries.

1 Introduction

It is well known that energy spectrum of the hydrogen atom had been calculated long begore creating the comprehensive quantum mechanical theory: Bohr [1, 2, 3], Sommerfeld [4, 5, 6, 7], Wilson 1915-Wilson(1), 1922-Wilson(2), Ishiwara [10], Planck [11, 12], Schwarzschild [13], Epstein [14], Wentzel [15], Brillouin [16, 17]. It was established that the Bohr-Sommerfeld rules, basis of the "old" quantum mechanics even without rigorous mathematical foundation, are closely related to the so-called WKB-approximation in the consistent quantum theory: see in Langer [18], Titchmarsh [19], Ponomarev [20].

Looking for exactly solvable models in the framework of "new" quantum theory, some coolness towards approximate (all the more without foundation) methods and any achievements of the Bohr-Sommerfeld mechanics was inevitable. But the same question arises in the literature: why in the case of hydrogen atom the Bohr-Sommerfeld rule leads to the known exact energy spectrum. Also, from time to time in the literature one can face the statement of the sort: in a potential φ the Bohr-Sommerfeld quantization gives an exact result $\epsilon_n(\varphi)$: Bailey [21], Froman and Froman [22], Krieger [23], Rosenzweig and Krieger [24], Nisio [25], Elutin and Krivchenkov [26], Voros [27, 28, 28, 30], De Witt and Morette [31], Neveu [32], Gomes et al [33], Dutt et al [34], Lemos and Natividade [35], Schopf [36], Katayama [37], Kobylinsky er al [38], Fujii and Funahashi [39], Robnik and Salasnich [40, 41], Delabaere et al [42], Kudryashov and Vanne [43].

In the present work we turn to an oscillator problem but now placed concurrently in three different curved space backgrounds: Euclid E_3 (zero curvature), Lobachevsky (negative constant curvature) H_3, and Riemann S_3 (positive constant curvature). We have considered Schródinger's equation. It is shown that

there can be constructed special WKB-series that provide us with exact spectra by taking into account only two firs terms of these series, in all three models E_3, H_3, S_3. This work continues two earlier considerations [44] and [45] of the analogous problem for hydrogen atom in space models E_3, H_3, S_3, motivation and general mathematical techniques are similar.

2 Oscillator in E_3 and WKB-quantization

Let us consider a non-relativistic oscillator in Euclid space model E_3. In Schrödinger's equation, the variables are separated by the known substitution $\Psi(r,\theta,\phi) = f(r)\, Y_{lm}(\theta,\phi)$:

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} + \left[\frac{2M}{\hbar^2}\,(E - \frac{1}{2}k\, r^2) - \frac{l(l+1)}{r^2} \right] f(r) = 0 \; . \qquad (2.1)$$

Let t be a new variable: $t:\; r^2 = e^{2t}$, eq. (2.1) takes the form

$$(\frac{d^2}{dt^2} + \frac{d}{dt})\; R + \left[\frac{2M}{\hbar^2}\,(Ee^{2t} - \frac{1}{2}ke^{4t}) - l(l+1) \right] f(r) = 0 \; .$$

Excluding the first derivative term by a substitution $R = e^{-t/2}\, S(t)$

$$\frac{d^2}{dt^2}S + \left[\frac{2MEe^{2t} - Mk\, e^{4t} - \hbar^2 l(l+1)}{\hbar^2} - \frac{1}{4} \right] S = 0 \; . \qquad (2.2)$$

With the notation

$$-\hbar^2 l(l+1) = -\hbar^2\, [(l+1/2)^2 - \frac{1}{4}] = -L^2 + \frac{\hbar^2}{4}\; , \quad A = -Mk \; ,$$
$$B = 2ME\; , \quad C = -L^2\; , \quad \Pi^2(t) = A\, e^{4t} + B\, e^{2t} + C$$

from eq. (2.2) we get

$$\frac{d^2}{dt^2}\, S(t) + \frac{\Pi^2(t)}{\hbar^2}\, S(t) = 0 \; . \qquad (2.3)$$

Now let us expand the function $Q(t)$ into a series in terms of $(\hbar/i)^n$

$$S(t) = \exp\,[\, \frac{i}{\hbar} \int Q(t)dt\,]\; ,$$
$$\frac{\hbar}{i}\,\frac{d}{dt}Q + Q^2 - \Pi^2 = 0\; , \quad Q(t) = \sum_{n=0}^{\infty} [\, (\frac{\hbar}{i})^n\, Q_n(t)\,]\; ,$$
$$Q_0 = \sqrt{\Pi^2}\; , \; Q_1 = -\frac{1}{2Q_0}\, Q_0'\; , \quad Q_2 = -\frac{1}{2Q_0}\, (\, Q_1' + Q_1^2\,)\; ,$$
$$Q_n = -\frac{1}{2Q_0}\, (\, \frac{d}{dt}Q_{n-1} + \sum_{k=1}^{n-1} Q_{n-k}Q_k\,) = 0\; , \; n = 3,4,5,...; \qquad (2.4)$$

the symbol $'$ denotes derivative d/dt.

We will assume that the wave $S(r)$ corresponding to a bound state, being considered as a function of complex variable t, has a finite number of of zeros in the complex plane, which are allocated at real axis between classical turning pints. According the known theorem in complex variable function theory the number of such zeros of $S(t)$ within certain domain can be calculated through derivative $(\ln S(t))'$ along a contour bounding that domain

$$\frac{1}{2\pi i} \oint_{\mathcal{L}} [\frac{d}{dt} \ln S(t)]\, dt = n \ ,$$

from whence substituting a series instead of $Q(t)$, we arrive at

$$\sum_{n=0}^{\infty} [\ (\frac{\hbar}{i})^n \oint_{\mathcal{L}} Q_n(t)\ dt\ ]\ = 2\pi\hbar\ n\ . \tag{2.5}$$

It should be especially emphasized that relationship (2.5) is a precise mathematical condition without any approximation. Accounting for only two first terms leads to the Bohr-Sommerfeld quantization rule

$$\oint_{\mathcal{L}} Q_0(t)dt + \frac{\hbar}{i} \oint_{\mathcal{L}} Q_0(t)dt \approx 2\pi\hbar\ n\ . \tag{2.6}$$

Calculation of the integrals is reduced to finding residues in two points

$$\oint_{\mathcal{L}} \frac{Q_n(z)}{z}\ dz\ = (-2\pi i) \sum \text{res}_{\ z=0,\infty} \frac{Q_n(z)}{z}\ . \tag{2.7}$$

The contribution of the first order term is

$$\oint_{\mathcal{L}} Q_0(t)\ dt = 2\pi\ (\ -i\sqrt{C} - i\ \frac{B}{2\sqrt{A}}\ )\ . \tag{2.8}$$

The contribution of the second order term is

$$\frac{\hbar}{i} \oint_{\mathcal{L}} Q_1(t)dt = \frac{\hbar}{i}\ (-\frac{1}{2})\ (-2\pi i)\ \times$$
$$\times \sum \text{res}_{\ z=0,\infty}\ \frac{1}{2}\ \frac{4Az^4 + 2Bz^2}{z(Az^4 + Bz^2 + C)} = -2\pi\hbar\ .$$

Therefore, the quantization rule (2.6) gives

$$2\pi\ (-i\sqrt{C} - i\ \frac{B}{2\sqrt{A}} - \hbar) = 2\pi\ (2n)\ ,$$

from whence it follows the exact energy spectrum

$$E = \hbar \sqrt{\frac{k}{M}}\ (2n + l + 3/2)\ . \tag{2.9}$$

3 Oscillator in hyperbolic model H_3

In Lobachevsky space, the Scrödinger equation for an oscillator problem

$$(-\frac{\hbar^2}{2M}\ \Delta_2\ +\frac{1}{2}k\rho^2 \text{th}^2\ r\ )\ \Psi = E\ \Psi$$

after separation of the variables $\Psi(r,\theta,\phi) = f(r)\ Y_{lm}(\theta,\phi)$ leads to

$$\frac{d^2 f}{dr^2} + \frac{2}{\text{th}\ r}\frac{df}{dr} + \left[\frac{2M\rho^2}{\hbar^2}(E - \frac{1}{2}k\rho^2 \text{th}^2 r) - \frac{l(l+1)}{\text{sh}^2 r}\right] f = 0\ . \qquad (3.1)$$

It should ne noted special symmetry property of the radial equation with respect to the change $r \to -r$. Negative values for r are non-physical, but below in taking into account zeros of complex variable function we must take into consideration these non-physical zeros as well.

Let t be a new variable: $\text{th}^2\ r = e^{2t}$, eq. (3.1) takes the form

$$(\frac{d^2}{dt^2} + \frac{d}{dt})f + \frac{1}{(1-e^{2t})^2}\left[\frac{2M\rho^2}{\hbar^2}(Ee^{2t} - \frac{1}{2}k\rho^2 e^{4t}) - l(l+1)(1-e^{2t})\right] f = 0\ .$$

Excluding the first derivative term by a substitution $f = e^{-t/2}\ S(t)$

$$\frac{d^2}{dt^2}S + \left[\ \frac{2ME\rho^2\ e^{2t} - Mk\rho^4\ e^{4t} - \hbar^2 l(l+1)\ (1-e^{2t})}{\hbar^2\ (1-e^{2t})^2} - \frac{1}{4}\ \right]\ S = 0\ . \quad (3.2)$$

Using the notation

$$-\hbar^2 l(l+1) = -\hbar^2\ [(l+1/2)^2 - \frac{1}{4}] = -L^2 + \frac{\hbar^2}{4}\ ,$$
$$A = -Mk\rho^4\ , \qquad B = 2ME\rho^2 - \hbar^2 + L^2\ , \qquad C = -L^2\ ,$$
$$\Pi^2(t) = \frac{A\ e^{4t} + B\ e^{2t} + C}{(1-e^{2t})^2}\ , \qquad \Delta(t) = \frac{5-e^{2t}}{4}\frac{e^{2t}}{(1-e^{2t})^2}\ , \qquad (3.3)$$

we arrive at

$$\frac{d^2}{dt^2}\ S(t) + \ [\ \frac{\Pi^2(t)}{\hbar^2}\ +\ \Delta(t)\ ]\ S(t) = 0\ . \qquad (3.4)$$

Further we follow the standard procedure

$$S(t) = \exp\ [\ \frac{i}{\hbar}\ \int\ Q(t)dt\ ]\ ,$$
$$\frac{\hbar}{i}\ \frac{d}{dt}Q + Q^2 - \Pi^2 + (\frac{\hbar}{i})^2\Delta = 0\ , \ \ Q(t) = \sum_{n=0}^{\infty}\ [\ (\frac{\hbar}{i})^n\ Q_n(t)\ ]\ ,$$
$$Q_0 = \sqrt{\Pi^2}\ , \ Q_1 = -\frac{1}{2Q_0}\ Q_0'\ , \ \ Q_2 = -\frac{1}{2Q_0}\ [\ Q_1' + Q_1^2 + \Delta\ ]\ ,$$
$$Q_n = -\ \frac{1}{2Q_0}\ [\ \frac{d}{dt}Q_{n-1}\ +\ \sum_{k=1}^{n-1}\ Q_{n-k}Q_k\ ] = 0\ , \ n = 3,4,5,... \qquad (3.5)$$

The quantization condition (exact that) take the form

$$\frac{1}{2\pi i} \oint_{\mathcal{L}} [\frac{d}{dt} \ln S(t)]\, dt = 2n$$

or

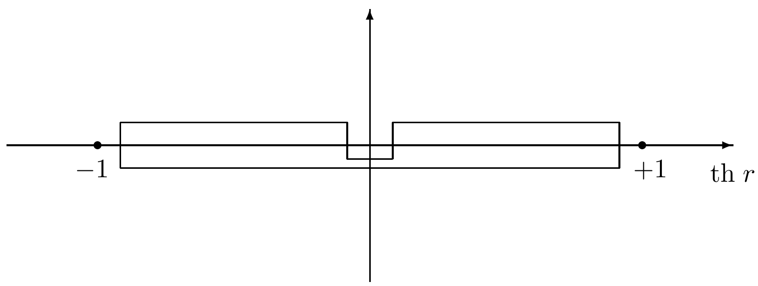
$$\sum_{n=0}^{\infty} [\ (\frac{\hbar}{i})^n \oint_{\mathcal{L}} Q_n(t)\ dt\ ] \ = 2\pi\hbar\ (2n)\ . \tag{3.6}$$

In particular, allowing for only two first terms of the WKB-series gives

$$\oint_{\mathcal{L}} Q_0(t) dt + \frac{\hbar}{i} \oint_{\mathcal{L}} Q_0(t) dt \approx 2\pi\hbar\ (2n)\ . \tag{3.7}$$

In calculating the contour integrals one is to use the variable $z = e^t =$ th r, correspondingly the contour $\mathcal{L}(z)$ including zeros of the function can be presented by the Figure

Fig. 1 (Integration contour $\mathcal{L}(z)$)



and we are to find residues in four points

$$\oint_{\mathcal{L}} \frac{Q_n(z)}{z}\, dz \ = (-2\pi i) \sum \text{res}_{\ z=0,\pm 1,\infty} \frac{Q_n(z)}{z}\ . \tag{3.8}$$

The contribution of the first order term is

$$\oint_{\mathcal{L}} Q_0(t)\ dt = 2\pi\ (\ -i\sqrt{C} + i\sqrt{A+B+C} + i\sqrt{A}\ )\ .$$

The contribution of the second order term is

$$\frac{\hbar}{i} \oint_{\mathcal{L}} Q_1(t) dt = \frac{\hbar}{i}\ (-\frac{1}{2})\ (-2\pi i)\ \times$$

$$\times \sum \text{res}_{\ z=0,\pm 1,\infty} \left[\frac{1}{2}\ \frac{4Az^4 + 2Bz^2}{z(Az^4 + Bz^2 + C)} + \frac{2z^2}{z(1-z^2)} \right]\ ;$$

therefore we get

$$\frac{\hbar}{i} \oint_{\mathcal{L}} Q_1(t) dt = \frac{\hbar}{i}\ (-\frac{1}{2})(-2\pi i)\ (-2) = -2\pi\hbar\ .$$

Thus, the Bohr-Sommerfeld rule gives

$$-\sqrt{-C}+\sqrt{-A-B-C}+\sqrt{-A}-\hbar\approx\hbar\ (2n)\ , \tag{3.9}$$

from whence it follows

$$\sqrt{Mk\rho^4-2ME\rho^2+\hbar^2}=\hbar(2n+l+\frac{3}{2})-\sqrt{Mk\rho^4}\ .$$

However, we know an exact quantization condition (from exact solution of the differential equation (3.1) in hypergeometric functions – see for example [37])

$$\sqrt{Mk\rho^4-2ME\rho^2+\hbar^2}=\hbar\ (2n+l+\frac{3}{2})-\sqrt{Mk\rho^4+\frac{\hbar^2}{4}}\ , \tag{3.10}$$

below it will be more convenient to use dimensionless form

$$=+\sqrt{\mu-2\epsilon+1}=2n+l+\frac{3}{2}-\frac{\sqrt{1+4\mu}}{2}\ . \tag{3.11}$$

It is the point of primary importance. Turning back to starting equation (3.2), one may note that from the very beginning in this equation a special formal rearrangement should be performed

$$\begin{gathered}\frac{d^2}{dt^2}S-\frac{1}{4}\ S+\frac{1}{\hbar^2(1-e^{2t})^2}\Big[\ (2ME\rho^2+\hbar^2\beta-\hbar^2\beta)+e^{2t}\\ +(-Mk\rho^4+\hbar^2\alpha-\hbar^2\alpha)e^{4t}-\hbar^2 l(l+1)(1-e^{2t})\ \Big]\ S=0\ ,\\ A=-Mk\rho^4+\hbar^2\ \alpha\ ,\qquad B=2ME\rho^2+\hbar^2\ \beta+\text{Ł}^2\ ,\\ C=-L^2\ ,\qquad \alpha=-\frac{1}{4}\ ,\qquad \beta=\frac{5}{4}\ ,\qquad \alpha+\beta=1\ ,\end{gathered}$$

which should give different representation of the main differential equation

$$\begin{gathered}\frac{d^2}{dt^2}S+\ \Big[\frac{(2ME\rho^2+\hbar^2\beta)e^{2t}+(-Mk\rho^4+\hbar^2\alpha)e^{4t}-\hbar^2 l(l+1)(1-e^{2t})}{\hbar^2(1-e^{2t})^2}\\ -\frac{1}{4}-\frac{5}{4}\frac{e^{2t}}{(1-e^{2t})^2}+\frac{1}{4}\frac{e^{4t}}{(1-e^{2t})^2}\ \Big]\ S=0\ .\end{gathered} \tag{3.12}$$

Correspondingly, we have expressions for A,B,C and $\Delta(t)$

$$\begin{gathered}\Pi^2(t)=\frac{A\ e^{4t}+B\ e^{2t}+C}{(1-e^{2t})^2},\ \ \Delta=-\frac{1}{4}-\frac{5}{4}\ \frac{e^{2t}}{(1-e^{2t})^2}+\frac{1}{4}\ \frac{e^{4t}}{(1-e^{2t})^2}\ ,\\ A=-Mk\rho^4-\frac{1}{4}\hbar^2\ ,\qquad B=2ME\rho^2-\frac{5}{4}\hbar^2+L^2\ ,\qquad C=-L^2\ ;\end{gathered} \tag{3.13}$$

from whence the Bohr-Sommerfeld rule results the exact energy spectrum (let $l+\frac{3}{2}+2n=N$)

$$2\epsilon = -N^2 + \sqrt{1+4\mu}\; N + \frac{3}{4}\;, \tag{3.14}$$

or differently

$$2\epsilon - 1 = +\mu - (N - \frac{\sqrt{1+4\mu}}{2})^2 \quad . \tag{3.15}$$

4 Oscillator in spherical space S_3

In Riemann spherical space , the Scrödinger equation for an oscillator problem

$$(-\ \frac{\hbar^2}{2M}\ \Delta_2\ + \frac{1}{2}k\rho^2 \mathrm{tg}^2\ r\ )\ \Psi = E\ \Psi$$

after separation of the variables $\Psi(r,\theta,\phi) = f(r)Y_{lm}(\theta,\phi)$ gives

$$\frac{d^2 f}{dr^2} + \frac{2}{\mathrm{tg}\ r}\frac{df}{dr} + \left[\frac{2M\rho^2}{\hbar^2}(E - \frac{1}{2}k\rho^2\mathrm{tg}^2\ r) - \frac{l(l+1)}{\sin^2 r} \right] f = 0\ . \tag{4.1}$$

Again, take notice symmetry $r \to -r$. Let r be a new variable: $\mathrm{tg}^2\ r = e^{2t}$, eq. (4.1) takes the form (let if be $f = e^{-t/2} S(t)$):

$$\frac{d^2}{dt^2}S + \left[\frac{2ME\rho^2 e^{2t} - Mk\rho^4\ e^{4t} - \hbar^2 l(l+1)(1+e^{2t})}{\hbar^2(1+e^{2t})^2} - \frac{1}{4} \right] S = 0\ . \tag{4.2}$$

With the notation

$$\begin{gathered} -\ \hbar^2 l(l+1) = -\hbar^2\ [(l+1/2)^2 - \frac{1}{4}] = -L^2 + \frac{\hbar^2}{4}\ , \\ A = -Mk\rho^4\ , \qquad B = 2ME\rho^2 + \hbar^2 - L^2\ , \qquad C = -L^2\ , \\ \Pi^2(t) = \frac{A\ e^{4t} + B\ e^{2t} + C}{(1+e^{2t})^2}\ , \qquad \Delta(t) = -\frac{5+e^{2t}}{4}\ \frac{e^{2t}}{(1+e^{2t})^2} \end{gathered} \tag{4.3}$$

eq. (4.2) reads

$$\frac{d^2}{dt^2}\ S(t) +\ [\ \frac{\Pi^2(t)}{\hbar^2}\ +\ \Delta(t)\ ]\ S(t) = 0\ . \tag{4.4}$$

Further we follow the above procedure

$$\frac{1}{2\pi i}\ \oint_{\mathcal{L}} [\frac{d}{dt}\ln S(t)]\ dt = 2n$$

or

$$\sum_{n=0}^{\infty} [\ (\frac{\hbar}{i})^n\ \oint_{\mathcal{L}}\ Q_n(t)\ dt\ ]\ = 2\pi\hbar\ (2n)\ . \tag{4.5}$$

The Bohr-Sommerfeld rule reads

$$\oint_{\mathcal{L}} Q_0(t)dt + \frac{\hbar}{i} \oint_{\mathcal{L}} Q_0(t)dt \approx 2\pi\hbar \ (2n) \ . \tag{4.6}$$

Calculating the contour integrals is reduced to calculating residues in four points in complex plane ($z = e^t = \text{tg}\ r$)

$$\oint_{\mathcal{L}} \frac{Q_n(z)}{z} \ dz \ = (-2\pi i) \ \sum \ \text{res}_{\ z=0,\pm i,\infty} \ \frac{Q_n(z)}{z} \ . \tag{4.7}$$

The Bohr-Sommerfeld rule leads to

$$-\sqrt{-C} + \sqrt{-A+B-C} - \sqrt{-A} - \hbar \approx \hbar \ (2n) \ .$$

and further we arrive at the relationship

$$\sqrt{Mk\rho^4 + 2ME\rho^2 + \hbar^2} = \hbar(2n + l + \frac{3}{2}) + \sqrt{Mk\rho^4} \ . \tag{4.8}$$

However, we know an exact quantization condition (from exact solution of the differential equation (3.1) in hypergeometric functions)

$$\sqrt{Mk\rho^4 + 2ME\rho^2 + \hbar^2} = \hbar \ (2n + l + \frac{3}{2}) + \sqrt{Mk\rho^4 + \frac{\hbar^2}{4}} \tag{4.9}$$

or in dimensionless notation

$$\sqrt{\mu + 2\epsilon + 1} = 2n + l + \frac{3}{2} + \frac{\sqrt{1+4\mu}}{2} \ . \tag{4.10}$$

Turning back to starting equation (4.2), one may note that from the very beginning in this equation a special formal rearrangement should be performed

$$\frac{d^2}{dt^2}S - \frac{1}{4}\ S + \frac{1}{\hbar^2(1+e^{2t})^2} \ \big[\ (2ME\rho^2 + \hbar^2\beta - \hbar^2\beta)e^{2t}$$
$$+(-Mk\rho^4 + \hbar^2\alpha - \hbar^2\alpha)e^{4t} - \hbar^2 l(l+1)(1+e^{2t})\big] \ S = 0,$$

$$A = -Mk\rho^4 - \hbar^2 \ \alpha \ , \qquad B = 2ME\rho^2 + \hbar^2 \ \beta - \text{Ł}^2 \ ,$$
$$C = -L^2 \ , \ \ \alpha = -\frac{1}{4} \ , \ \beta = \frac{5}{4} \ , \ \ \alpha + \beta = 1 \ ,$$

which should give different representation of differential equation

$$\frac{d^2}{dt^2}S + \ \Big[\frac{(2ME\rho^2 + \hbar^2\beta)e^{2t} + (-Mk\rho^4 - \hbar^2\alpha)e^{4t} - \hbar^2 l(l+1)(1+e^{2t})}{\hbar^2(1+e^{2t})^2}$$
$$-\frac{1}{4} - \frac{5}{4}\ \frac{e^{2t}}{(1+e^{2t})^2} - \frac{1}{4}\ \frac{e^{4t}}{(1+e^{2t})^2}\ \Big]\ S = 0 \ . \tag{4.11}$$

so that A, B, C and $\Delta(t)$ are

$$\Pi^2(t) = \frac{A\ e^{4t} + B\ e^{2t} + C}{(1-e^{2t})^2}\ , \quad \Delta = -\frac{1}{4} - \frac{5}{4}\ \frac{e^{2t}}{(1+e^{2t})^2} - \frac{1}{4}\ \frac{e^{4t}}{(1+e^{2t})^2}\ .$$
$$A = -Mk\rho^4 + \frac{1}{4}\hbar^2\ , \qquad B = 2ME\rho^2 + \frac{5}{4}\hbar^2 - L^2\ , \qquad C = -L^2\ ; \tag{4.12}$$

from whence the Bohr-Sommerfeld rule results the exact energy spectrum (let $l + \frac{3}{2} + 2n = N$)

$$2\epsilon = (l + 2n + \frac{3}{2})^2 + \sqrt{1+4\mu}\ (l + 2n + \frac{3}{2}) - \frac{3}{4}\ . \tag{4.13}$$

or differently

$$2\epsilon + 1 = \mu + (N + \frac{\sqrt{1+4\mu}}{2})^2\ . \tag{4.14}$$

Let summarize result:

It is shown that in the spaces of Lobachevsky and Riemann, similar to the case od Euclidean model E_3, in WKB-theory for harmonic oscillator potential there can be constructed special WKB-series, such that only two first terms give a non-zero contribution into the Bohr – Sommerfeld quantization condition providing us with an exact energy spectrums in all three models E_3, H_3, S_3.

5 Acknowledgement

The work was supported by the grand of BRFFI (Belarusian Republican Foundation for Fundamental Research), No F09K–123.

We wish to thank the Organizers of the International Conference "Non-Euclidean Geometry and its applications." (5 – 9 July 2010, Kluj-Napoca (Kolozvár), ROMANIA) for having given us the opportunity to talk on this subject, as well as HCAA-ESF (Harmonic and Complex Analysis and its Applications – European Science Foundation Research Networking Programme) for partial support.

References

[1] N. Bohr. On the Constitution of atoms and molecules // Phil. Mag. 1913. Vol. 26. P. 1 – 25, 476 – 502, 857 – 875

[2] N. Bohr. Effet des champsélectrique et magnètique sur les raies spectrales // Phil. Mag. 1914. Vol. 27. P. 506.

[3] N. Bohr. Sur les séries spectrales de l'hydrogène et la structure de l 'atome. // Phil. Mag. 1915. Vol. 29. P. 332.

[4] A. Sommerfeld. Zur Theorie der Balmerschen Serie // Münchener Berichte. 1915. S. 425 – 458.

[5] A. Sommerfeld. Die Feinstructur der wasserstoff und wasserstoffähnlichen Linien // Münchener Berichte. 1915. S. 459 – 500.

[6] A. Sommerfeld. Zur Quantentheorie der Spektrallinien // Annalen der Physik. 1916. Bd. 51. S. 1 – 94, S. 125 – 167.

[7] A. Sommerfeld. Atombau und Spektrallinien. Braunchweig, Vieweg, 1919, S. 327 – 357; 520 – 522; Atomic structure and spectral lines. L.: Methuen, 1923, P. 467 – 496; 608 – 611.

[8] W. Wilson. The quantum theory of radiation and line spectra // Phil. Mag. 1915. Vol. 29. P. 795 – 802.

[9] W. Wilson. The quantum theory and electromagnetic phenomena // Proc. Roy. Soc. London. A. 1922. Vol. 102. P. 478 – 483.

[10] J. Ishiwara. Die universelle Bedeutung des Wirkungsquantums. // Tokio Sugaku Buturigakkawi Kizi. 1915. Bd 8. S. 106 – 116.

[11] M. Planck. Die Quantenhypothese für Molekeln mit mehreren Freiheitsgraden // Verhandlungen der Deutschen Physikalischen Gesellschaft. 1915. Bd. 17. S. 407 – 418; S. 438 – 451.

[12] M. Planck. Die physikalische Struktur des Phasenraum // Annalen der Physik. 1916. Bd. 50. S. 385 – 418.

[13] K. Schwarzschild. Zur Quantenhypothese // Berliner Berichte. 1916. S. 548 – 568.

[14] P.S. Epstein. Zur Theorie des Starkeffektes // Phys. Zeit. 1916. Bd. 17. S. 148 – 150.

[15] G. Wentzel. Eine Verallgemeinerung der Quantembedingungen für die Zwecke der Wellenmechanik // Zeit. Phys. 1926. Bd. 38. S. 518 – 529.

[16] L. Brillouin. Théorie des quanta et l'atome de Bohr. Press Universitaire de France. 1922.

[17] L. Brillouin. La méchanique ondulatoire de Schrödinger; une méthode générale de résolution par approximations successives // Coptes Rendus Acad. Sci. Paris. 1926. T. 183. P. 24-44.

[18] R.E. Langer. On the connection formulas and the solutions of the wave equation // Phys. Rev. 1937. Vol. 51. P. 669 – 676.

[19] E.C. Titchmarsh. On the asymptotic distridution of eigenvalues // Quart. J. Math. 1954. Vol. 5. P. 228 – 240.

[20] L.I. Ponomarev. Lectures in quasiclassics. Kiev, 1966 (in Russian).

[21] P.B. Bailey. Exact quantization rules for the one-dimensional Scrödinger equation with turning points // J. Math. Phys. 1964. Vol. 5, N 9. P. 1293 – 1297.

[22] N. Froman, P.O. Froman. JWKB Approximation: contributions to the theory. North-Holland Publ. Co. Amsterdam. 1965.

[23] J.B. Krieger. Asymptotic properties of perturbation theory // J. Math. Phys. 1966. Vol. 9, N 3. P. 432 – 435.

[24] C. Rosenzweig, J.B. Krieger. Exact quantization conditions // J. Math. Phys. 1968. Vol. 9, N 6. P. 849 – 860.

[25] Sigeko Nisio. The formation of the Sommerfeld quantum theory of 1916 // Jap. Stud. Hist. Sci. 1973, N 12. P. 54 – 60.

[26] P.V. Elutin, D.V. Krivchenkov. On applicabilitu os the quasiclasiccal approximation. // TMF. 1974. Vol. 19. № 2. P. 233 – 236 (in Russian).

[27] A. Voros. Semi-classical approximations // Ann. Inst. H. Poincaré. A. 1976, Vol. 24. P. 31 – 90.

[28] A. Voros. The return of the quadratic oscillator. The complex WKB method // Ann. Inst. H. Poincaré. A. 1983. Vol. 39. P. 211 – 338.

[29] A. Voros. Exact quantization condition for anharmonic oscillators (in one dimension) // J. Phys. A. 1994. Vol. 27. P. 4653 – 4661.

[30] A. Voros. Exercises in exact quantization // J. Phys. A. 2000. Vol. 33. P. 7423 – 7450.

[31] De Witt-Morette C. The semiclassical expansion. // Ann. Phys. N.Y. 1976. Vol. 97. P. 367 – 399.

[32] A. Neveu. Semiclassical quantization methods in field theory // Phys. Rep. C. 1976. Vol. 23. P. 265 – 272.

[33] M.A.F. Gomes, M.T. Thomaz, G.L. Vasconcelos. Matrix formulation for the Wentzel-Kramers-Brillouin quantization rule // Phys. Rev. A. 1986. Vol. 34, N 5. P. 3598 – 3604.

[34] R. Dutt, A. Khare, U.P. Sukhatme. Exactness of supersymmetric WKB spectra for shape-invariante potentials // Phys. Lett. B. 1986. Vol. 181. P. 295 – 298.

[35] N.A. Lemos, C.P. Natividade. Harmonic oscilator in expanding universes // Nuovo Cim. B. 1987. Vol. 99, N 2. P. 211 – 225.

[36] H.G. Schöpf. Zur Geschichte der Bohr-Sommerfeldschen Quantentheorie // Ann. Phys. 1988, Bd. 45, N 8. S. 595 – 604.

[37] N. Katayama. A note on a quantum-mechanical harmonic oscillator in a a space of constant curvature // Nuovo Cim. B. 1992. Vol. 107, N 7. P. 763 – 768.

[38] N.A. Kobylinsky, S.S. Stepanov, R.S. Tutik. Semiclassical approach to ground state within the Klein-Gordon equation // J. Phys. A. 1990. Vol. 23, N 6. P. 237 – 241.

[39] K. Fujii, K. Funahashi. Ecactness in the WKB-approximation for some homogeneous spaces // J. Math. Phys. 1995. Vol. 36. P. 4590 – 4611; hep-th/9501145.

[40] M. Robnik, L. Salasnich. WKB exactness for the angular momentum and the Kepler problem: from the torus quantization to the exact one // J. Phys. A. 1997. Vol. 30. P. 1719 – 1729; quant-ph/9603014.

[41] M. Robnik, L. Salasnich. WKB to all orders and the accuracy of the semiclassical quantization // J. Phys. A. 1997. Vol. 30. P. 1711 – 1718; quant-ph/9610027.

[42] E. Delabaere, H. Dillinger and F. Pham. Exact semiclassical expansions for one-dimensional quantum oscillators // J. Math. Phys. 1997. Vol. 38. P. 6126 – 6184.

[43] V.V. Kudryashov, Yu.V. Vanne. Explicit summation of the constituent WKB series and new approximate wave functions // J. Appl. Math. 2002. Vol. 2. P. 265 (2002); arXiv:quant-ph/0102111.

[44] V.S. Otchik V.M. Red'kov. Quantum-mechanical Kepler problem in spve of constant curvarure. Preprint 298, IF AN BSSR. Minsk, 1986. 49 pages (in Russian).

[45] V.M. Red'kov. On WKB-quantization in Lobachevski and Riemann 3-spaces. // Nonlinear phenomena in complex systems. 2003, Vol. 6, № 2. P. 654-668.